4. Probability

Random experiment

- a random experiment is a process or course of action, whose outcome is uncertain
- performing the same random experiment repeatedly may result in different outcomes, therefore, the best we can do is to talk about the *probability of occurrence of a certain outcome*
- to determine the probabilities we need to define the possible outcomes first, then build an *exhaustive* list of all possible outcomes and, finally, make sure the listed outcomes are *mutually exclusive*
- a list of outcomes that meets the two conditions above is called a *sample space*

Random experiment: terminology

A *sample space* of a random experiment is a list of all possible outcomes of the experiment. The outcomes must be mutually exclusive and exhaustive.

The individual outcomes are called *simple events* or just *outcomes*. Simple events cannot be further decomposed into constituent outcomes.

An *event* is any collection of one or more simple events.

Our objective is to determine the probability that a certain event will occur.

Probability space

Probability Space = Sample Space + Probability

Assigning probabilities

Given a sample space $S = \{E_1, E_2, \dots, E_n\}$, the following properties for the assigned probabilities $P(E_i)$ of the simple events E_i *must hold:*

1. $0 \le P(E_i) \le 1$ for every i

2. $P(E_1) + P(E_2) + \cdots + P(E_n) = 1$

If event A is consist of k outcomes $E_{i_1}, E_{i_2}, \dots, E_{i_k}$ then

$$P(A) = P\left(E_{i_1}\right) + P\left(E_{i_2}\right) + \cdots + P(E_{i_k})$$

Probability space: example

Example 1

Two coins are tossed, and their up faces are recorded. List all the simple events (all the outcomes) for this random experiment.

Solution

Since physically there are two distinct coins, we have the following list of all the outcomes:

$$HH, HT, TH, TT$$

For instance, "HH" means that we observe H on coin 1 and H on coin 2.

Probability of event

Classic Probability

Consider a finite sample space S such that each outcome (simple event) is equally likely to occur. Then in this special case the probability of event A is given by

$$P(A) = \frac{Number\ of\ outcomes\ favorable\ to\ the\ event\ A}{Total\ number\ of\ outcomes\ in\ S}$$

Probability space: example

Example 1 – cont'd

What is the probability of each outcome? What is the probability of the following event

$$A = \{Observe\ one\ head\ and\ one\ tail\}$$

Solution

Because of symmetry all the outcome are equally likely, and in total there are 4 outcomes. Therefore,

$$P(HH) = P(HT) = P(TH) = P(HH) = 1/4$$

Next, since

$$A = \{HT, TH\}$$

we get that

$$P(A) = P(HT) + P(TH) = 1/2$$

Event operations

Let A and B be two events, then

- the *union* of A and B:

$$A\ orB = A \cup B = \{A\ occurs\ or\ B\ occurs\ or\ both\}$$

- the *intersection* A of and B*:*

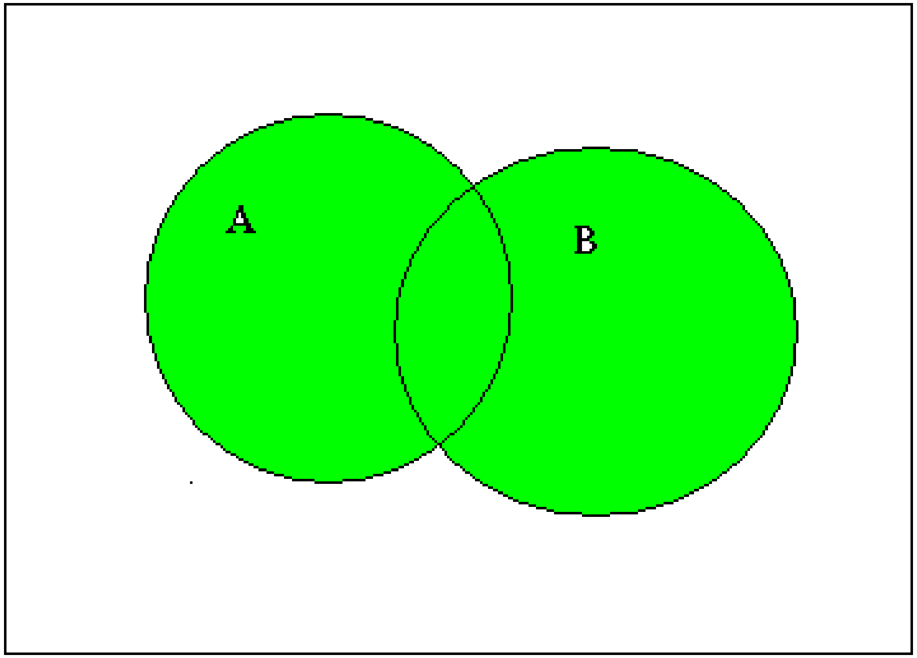
$$A\ and\ B = A \cap B = \{both\ A\ and\ B\ occur\}$$

- the *complement* of A*:*

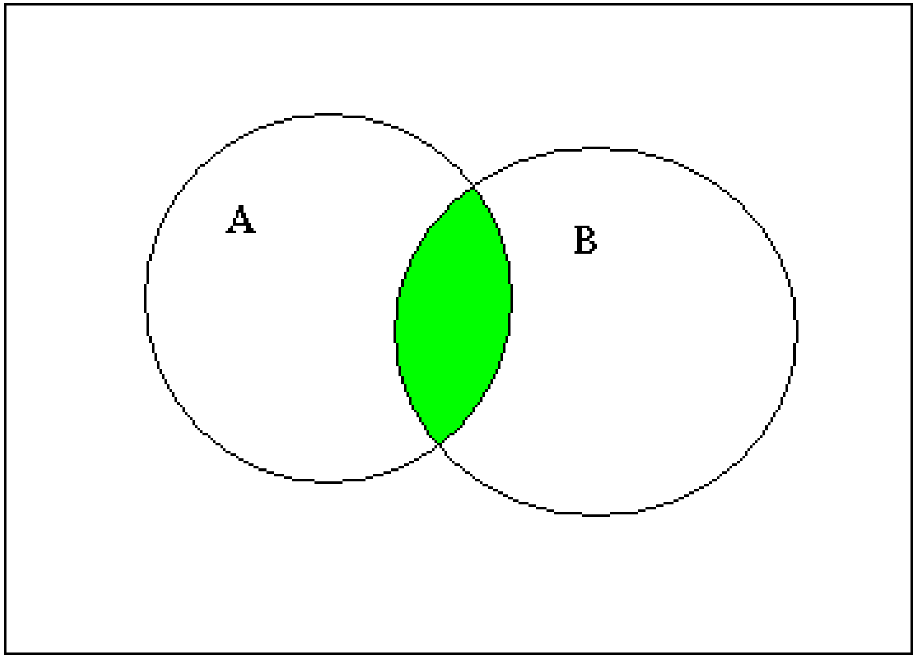
$$\overline{A} = A^c = \{A\ does\ not\ occur\}$$

Two events A and B are *mutually exclusive,* if they cannot occur at the same time, that is, their intersection is empty.

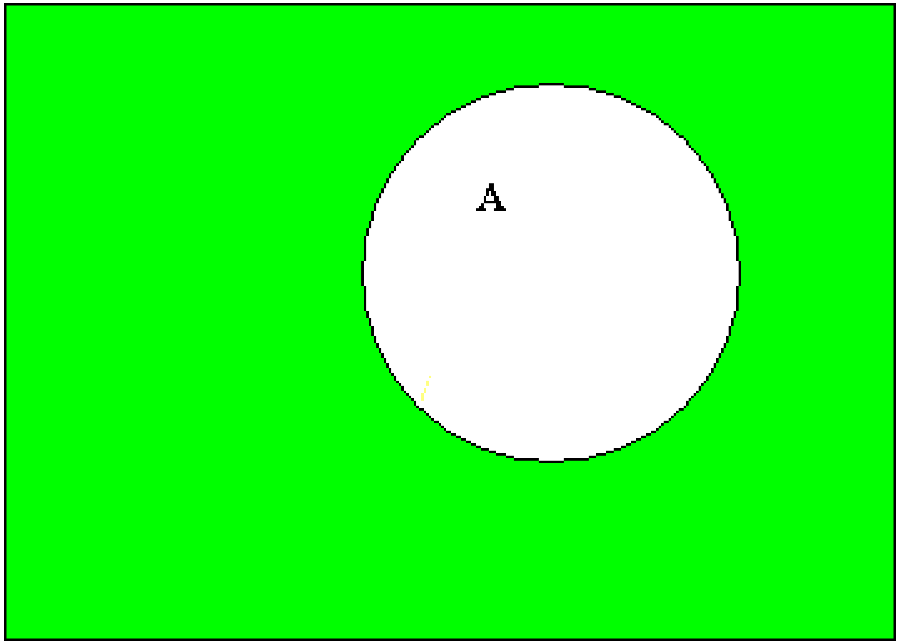
Venn Diagram: union



Venn Diagram: intersection



Venn Diagram: complement



Probability rules

Complement rule

Each outcome must belong to either A or complement of A. Since the sum of the probabilities assigned to all the outcomes is one, we have for any event A

$$P(A^c) = 1 - P(A)$$

Addition rule **(*or Inclusion-Exclusion Formula*)**

For any two events A and B

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

Probability rules: example

Example 2 Three coins are tossed. What is the probability of observing at least one tail?

Solution

Sample space: $\boldsymbol{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}$

Probability:

$$\boldsymbol{P(HHH) = P(HHT) = P(HTH) = P(THH) =}$$
$$\boldsymbol{P(TTH) = P(THT) + P(HTT) = P(TTT) = 1/8}$$

The event of observing at least one tail is the complement of the event of observing three heads. Hence,

$$\boldsymbol{P(observing\ at\ least\ one\ T) = 1 - P(HHH) = 1 - \frac{1}{8} = \frac{7}{8}}$$

Multiplication principle

Multiplication Principle. Suppose that a task is composed of two consecutive operations. If operation 1 can be performed in m ways and, for each of this, operation 2 can be performed in n ways, then the complete task can be performed in $m \times n$ ways.

Generalized Multiplication Principle. Suppose that a task consists of k operations performed consecutively. Suppose that operation 1 can be performed in m_1 ways; for each of these, operation 2 in m_2 ways; for each of these, operation 3 in m_3 ways; and so forth. Then the task can be performed in $m_1 \times m_2 \times \cdots \times m_k$ ways.

Multiplication principle: example

Multiplication Principle Exercise. How many license plates consisting of two letters followed by four digits are possible? How many license plates are possible if all the symbols must be different?

Draws with replacement

Exercise 1 **The box contains 3 blue and 2 red balls, well-mixed together. One ball is drawn randomly from the box and its color observed. The ball is placed back in the box, then the second ball is drawn. What is probability that both balls are blue?**

Draws without replacement

Exercise 2 **The box contains 3 blue and 2 red balls, well-mixed together. One ball is drawn randomly from the box without replacement and its color observed, then the second ball is drawn. What is probability that both balls are blue?**

Birthday paradox

Exercise 3 The problem is to compute the approximate probability that in a room of 30 people, at least two have the same birthday. For simplicity, we disregard variations in the distribution, such as leap years, twins, seasonal or weekday variations, and assume that the 365 possible birthdays are equally likely.