5. Conditional Probability and Independence

Conditional Probability

- The probability of an event when partial knowledge about the outcome of an experiment is known is called *conditional probability.*

- *The conditional probability that event A occurs given that event B has occurred* is denoted by $P(A|B)$, and it is given by this simple formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Example

Example 1. The following data are characteristics of the voting-age population regarding the 1992 presidential election in the United States. Number of persons is measured in thousands. For a randomly selected person from the population, let A be the event that the person selected voted, and B be the event that the person selected is a male. Find each of the following:

1. $P(A)$
2. $P(B)$
3. $P(A \text{ and } B)$
4. $P(A|B)$

Example – cont’d

	Voted	Did not vote	Total
Males	53,312	35,245	88,557
Females	60,554	36,573	97,127
Total	113,866	71,818	185,684

Solution

1. $P(A) = 113866/185684 \approx .613$

2. $P(B) = 88557/185684 \approx .477$

3. $P(A\ and\ B) = 53312/185684 \approx .287$

4. $P(A|B) = P(A\ and\ B)/P(B) = \frac{53312/185684}{88557/185684} = \frac{53312}{88557} \approx .602$

Independent and Dependent Events

Two events A and B are said to be *independent,* iff

$$P(A\ and\ B) = P(A)P(B)$$

Otherwise, the events are *dependent*.

Note that if two events are independent then the occurrence of one event does not change the likelihood of occurrence of the other event. Indeed,

$$P(A|B) = \frac{P(A\ and\ B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

Example – cont'd

Example 1 – cont'd

Are events A and B independent?

Solution

Since

$$\frac{53312}{185684} = P(A\ and\ B) \neq P(A)P(B) = \frac{113866}{185684} \times \frac{88557}{185684}$$

the events A and B are *not* independent.

Probability rules again

Complement rule: **For any event A**

$$\boldsymbol{P(A^c) = 1 - P(A)}$$

Addition rule: **For any two events A and B**

$$\boldsymbol{P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)}$$

Probability rules again

Multiplication rule: For any two events A and B

$$P(A \text{ and } B) = P(A) \times P(B|A) = P(B) \times P(A|B)$$

When A and B are independent

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example

Example 2. Let A and B be independent events with $P(A) = .3$ and $P(B) = .4$. What is $P(A\ or\ B)$?

Solution

By the addition law:

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B) = .7 - P(A\ and\ B)$$

Because of the independence of these events,

$$P(A\ and\ B) = P(A)P(B) = .3 \times .4 = .12$$

Therefore,

$$P(A\ or\ B) = .7 - .12 = .58$$

Probability Trees

This is a useful device to build a sample space and to calculate probabilities of simple events and events.

Rules for constructing a probability tree

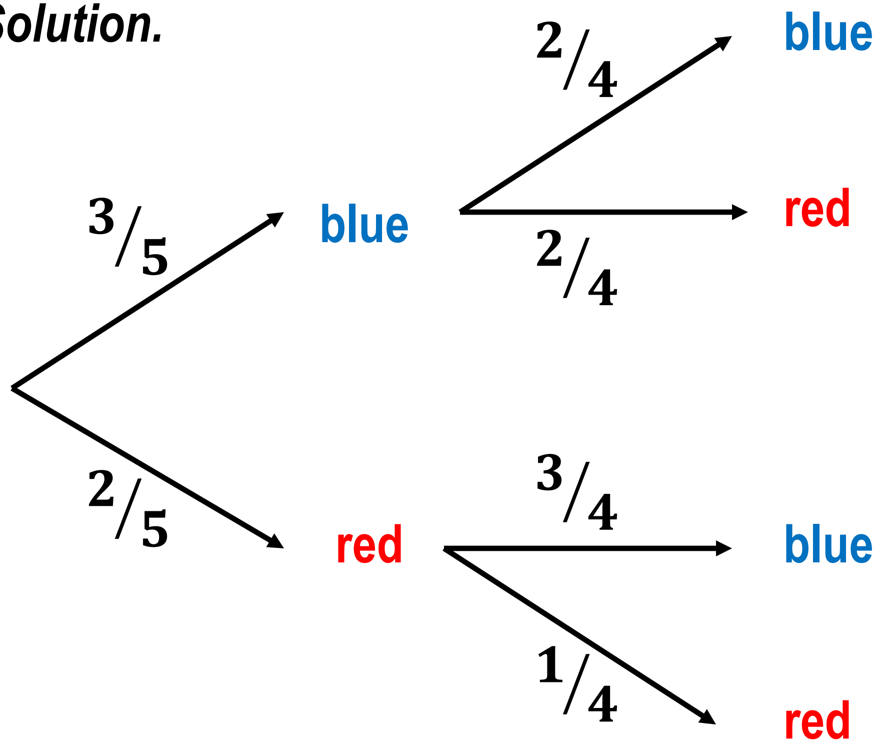
- Events forming the first set of branches must have known marginal probabilities, must be mutually exclusive, and should exhaust all possibilities (so that the sum of branch probabilities is 1)
- Events forming the second set of branches must be entered at the tip of each of the sets of first branches. Conditional probabilities, given the relevant first branch, must be entered, unless assumed independence allows the use of unconditional probabilities. Again, the branches must be mutually exclusive and exhaustive
- If there are additional sets of branches, the probabilities must be conditional on all preceding events. As always, the branches must be mutually exclusive and exhaustive.
- The sum of path probabilities must be taken over all paths included in the relevant event.

Probability tree: example

Example 4. The box contains 3 blue and 2 red balls, well-mixed together. One ball is drawn randomly from the box without replacement and its color observed, then the second ball is drawn. What is probability that both balls are blue? What is probability that one blue ball and one red ball are drawn?

Probability tree: example

Solution.



$$\frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$$

$$\frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$$

$$\frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$$

$$\frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$$

Probability tree: example

Denote the event when both drawn balls are blue by A. Then according to the probability tree there is only one path (outcome) favorable to the event. Therefore,

$$P(A) = \frac{6}{20} = .3$$

Denote the event when one blue ball and one red ball are drawn by B. There are two paths (outcomes) that are favorable to the event. Therefore, the probability of B is the sum of two corresponding probabilities:

$$P(B) = \frac{6}{20} + \frac{6}{20} = .6$$

Probability tree: exercises

Exercise 1. Suppose that the reliability of a test for hepatitis is specified as follows: Of people with hepatitis, 95% have a positive reaction and 5% have a negative reaction; of people free of hepatitis, 90% have a negative reaction and 10% have a positive reaction. From a large population of which .05% of the people have hepatitis, a person is selected at random and given the test. If the test is positive, what is the probability that the person actually has hepatitis?

Probability tree: exercises

Exercise 2. A factory has two machines that produce bolts. Machine I produces 60% of the daily output of bolts, and 3% of its bolts are defective. Machine II produces 40% of the daily output, and 2% of its bolts are defective.

1. What is the probability that a bolt selected at random will be defective?
2. If a bolt is selected at random and found to be defective, what is the probability that it was produced by machine I?

Probability tree: exercises

Exercise 3. Draw a tree diagram that illustrates the following. Three-fifths of kindergarten children are bussed to school, while two-fifths of the first to fifth graders are bussed. The school has grades K through 5, and 17.5% of the students are in kindergarten.

1. Determine the probability that a child chosen at random from the school is bussed to school.
2. If the child selected is bussed to school, what is the probability that the child is a kindergartener?

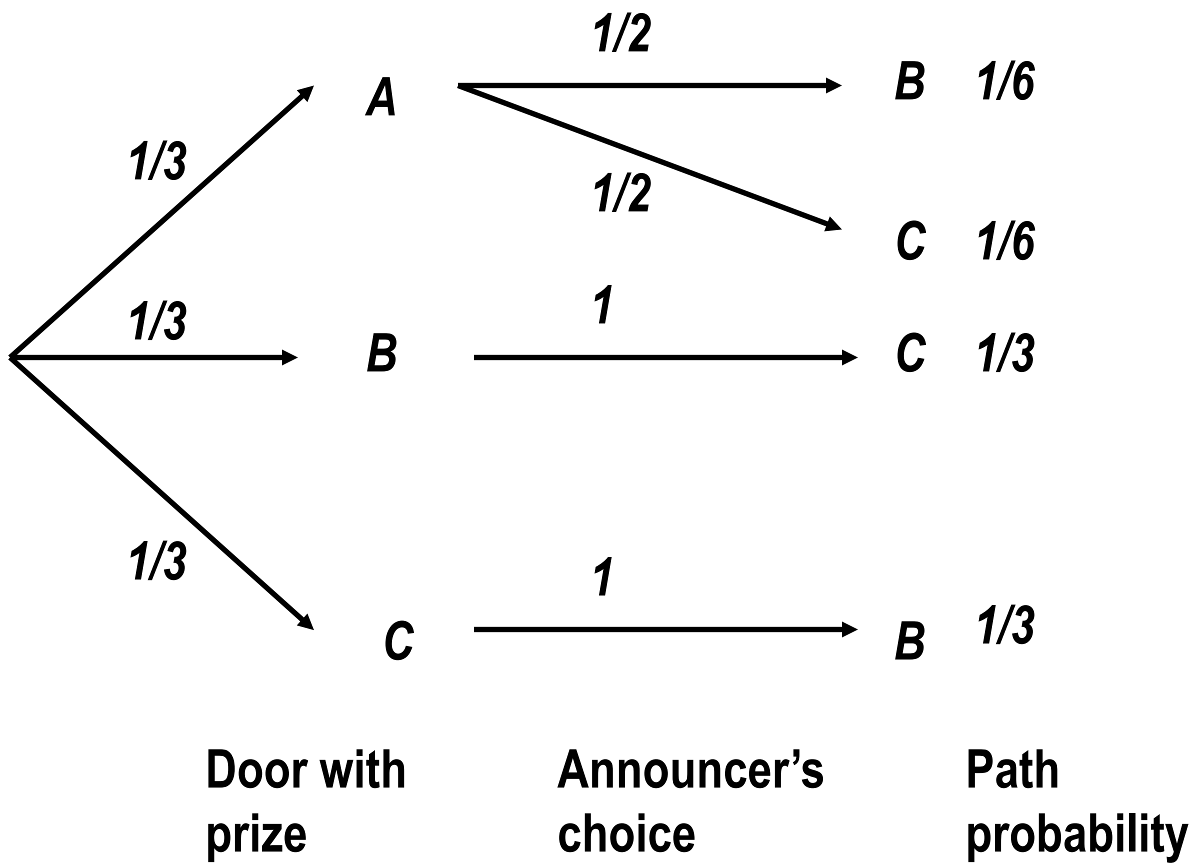
Probability tree: example

Example 5. In a certain television game show, a valuable prize is hidden behind one of the three doors. You, the contestant, pick one of the three doors. Before opening it, the announcer opens one of the other two doors and you see that the prize isn't behind that door. The announcer offers you the chance to switch the remaining door. Should you switch, or it does not matter?

Probability tree: example

Solution. Call the door that you select A, the others B and C. Assuming that the prize is distributed randomly among the doors, the probability that it's behind each of the doors is 1/3. If you picked a wrong door in A, the announcer has no choice. If B contains the prize, the announcer must open C; if C has the prize, he must open B. But if you picked correctly and A has the prize, the announcer does have a choice. Let us assume that the announcer picks B or C randomly, each with probability 1/2 in this situation. We can construct the following tree:

Probability tree: example



Probability tree: example

Suppose that the announcer has chosen B (and you chose A initially). What is the probability that the prize behind C?

$$P(\text{behind } C | \text{chose } B) = \frac{P(\text{behind } C \text{ and chose } B)}{P(\text{chose } B)}$$

$$= \frac{1/3}{1/6 + 1/3} = 2/3$$

So

$$P(\text{behind } A | \text{chose } B) = 1 - \frac{2}{3} = \frac{1}{3}$$

You have a better chance of winning if you switch to door C!